Raffles Mathematical Olympiad 2024
Round 1 (Solutions)

Date: 26 March 2024 **Duration: 1 hour**

This paper consists of 20 questions.
*For practice purpose, the multiple choice options are removed.

The marks allocation is as follows:

Question Number	Correct	Unanswered	Incorrect
1 to 10	4 marks	1 mark	0 mark
11 to 20	6 marks	1 mark	0 mark

1.

We have four cases to consider on the multiples of 8.

1°: There is only one single digit multiple of 8 and that is 8.

2°: For two-digits, the multiples are 16 $(=8\times 2)$, ..., 96 $(=8\times 12)$

There are $12-2+1=11$ such multiples and 22 digits.

3°: For three-digits, we have 104 $(=8\times 13)$, ..., 992 $(=8\times 124)$

There are 112 such multiples and 336 digits.

4°: For four-digits, we have 1000 $(=8\times 125)$, ..., 2024 $(=8\times 253)$

There are 129 such multiples and 516 digits.

Thus, there are $1+22+336+516=\boxed{875}$ digits in x.

2.

Let the number of boys and girls in 2023, respectively, be x and y.

$x=y+30$ ....(1)

$1.1(x+y)=1.05x+1.2y$ ....(2)

Subtituting (1) into (2):

$2.2y+33=1.05y+31.5+1.2y$

$0.05y=1.5$

$y=30$ and $x=60$

i.e. number of member this year $=1.1(30+60)=\boxed{99}$

3.

We can partition the 19 cards into three groups as follows:

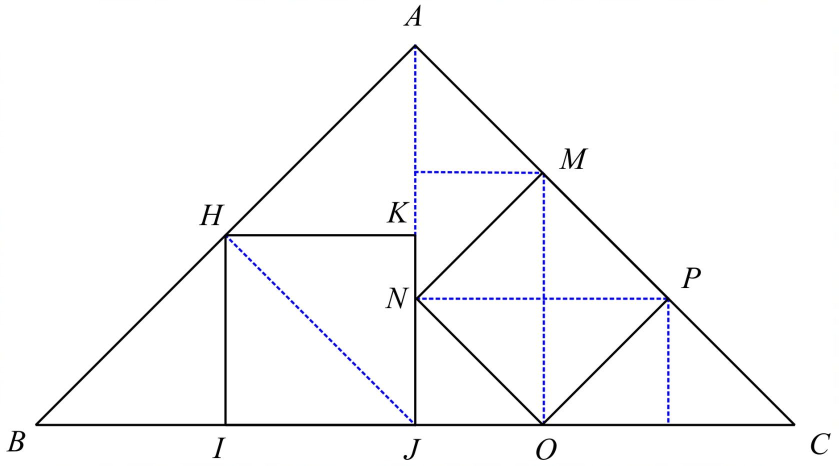
Group 1:	1	2	3	4	5	6	7	8	9
Group 2:	19	18	17	16	15	14	13	12	11
Group 3:	10								

For the choice of ten cards 1, 3, 5, 7, 9, 12, 14, 16, 18 and 10, there is no two that add up to 20.

If we choose 11 cards, then there are at least two cards that add up to 20.

Thus, the minimum number of cards to draw is $\boxed{11}$.

4.



We divide the triangle into two congruent isosceles right triangles: ΔABJ and ΔACJ. By drawing auxiliary lines as shown, we have four congruent triangles in ΔABJ and nine congruent triangles in ΔACJ.

Using $[ABC]$ to denote area of triangle ABC,

$$\begin{aligned}[MNOP] &= \frac{4}{9}\times[ACJ] \\ &= \frac{4}{9}\times 2[HIKJ] \\ &= \frac{8}{9}\times 126 \\ &= \boxed{112}\ \text{cm}^2\end{aligned}$$

5.

Suppose $2024 = (n+1)+(n+2)+\ldots+(n+k)$

$$= kn + \frac{1}{2}(k)(k+1)$$

Then $4048 = k(2n+k+1)$

We note that $k < 2n+k+1$ and $4048 = 2^4 \times 11 \times 23$.

To find greatest k, we try to get 4048 as a product of two factors which are as close as possible.

By considering cases in decreasing size of k,

1°: $k = 2\times 23$,

Then $2n+46+1 = 2^3\times 11 \Rightarrow 2n = 41$, is not possible.

2°: $k = 2^2 \times 11$

Then $2n+44+1 = 2^2\times 23 \Rightarrow 2n = 47$, is not possible.

3°: $k = 23$

Then $2n+23+1 = 2^4 \times 11 \Rightarrow 2n = 152$ is possible.

Thus, greatest $k = \boxed{23}$.

6.

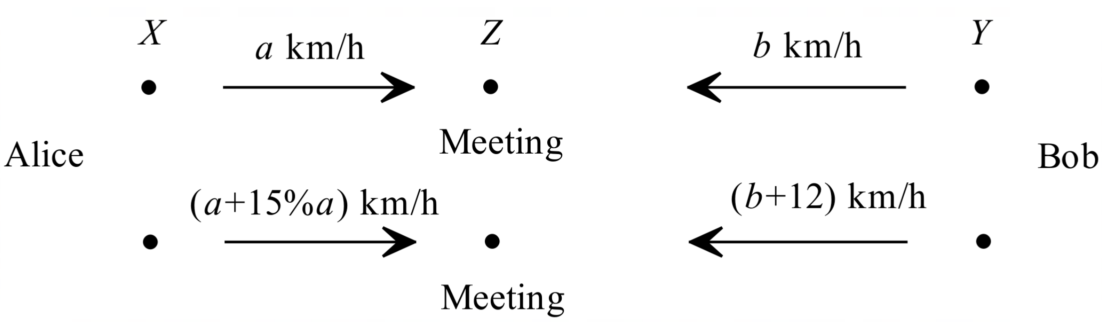
Sequence	Difference in successive terms
9,	
24,	$15 = 3 \times 5$
69,	$45 = 3^2 \times 5$
204,	$135 = 3^3 \times 5$
609,	$405 = 3^4 \times 5$
x,	
y, $\ldots$	

Thus, $x = 609 + 3 \times 405 = 609 + 1215 = 1824$

and $\quad y = 1824 + 3 \times 1215 = 5469$

and $x + y = \boxed{7293}$.

7. Let the original speed in km/h for Alice and Bob be a and b respectively.
Let the meeting be at Z.



Originally: $\dfrac{XZ}{a} = \dfrac{YZ}{b} \quad \Rightarrow \quad \dfrac{XZ}{YZ} = \dfrac{a}{b}$

After increase in speed: $\dfrac{XZ}{a + 15\%a} = \dfrac{YZ}{b + 12} \quad \Rightarrow \quad \dfrac{XZ}{YZ} = \dfrac{a + 15\%a}{b + 12}$

Thus, $\dfrac{a}{b} = \dfrac{a + 15\%a}{b + 12}$

$b + 12 = 1.15b$

$0.15b = 12$

$b = 80$

i.e. Bob's original speed was $\boxed{80 \text{ km/h}}$.

8.

Let V be the volume of each of the 3 bottles.

Volume of concentrate in first bottle $= \frac{2}{3}V$

" " " " second bottle $= \frac{4}{7}V$

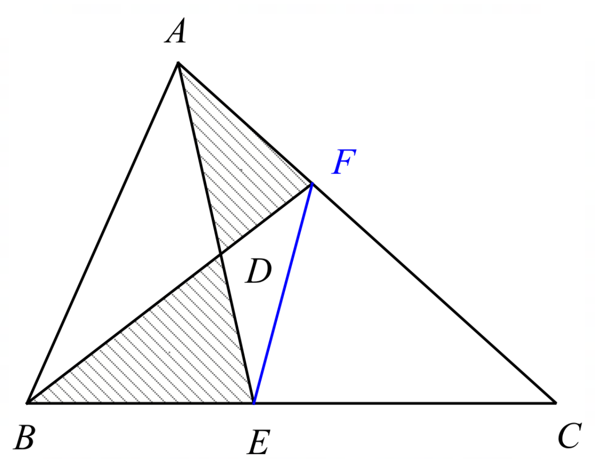
" " " " third bottle $= \frac{7}{12}V$

Note that the container holds $3V$ of fruit juice, of which

volume of concentrate in container $= \left(\frac{2}{3} + \frac{4}{7} + \frac{7}{12}\right)V = \frac{51}{28}V$

Thus, ratio $= \dfrac{\frac{51}{28}V}{3V - \frac{51}{28}V} = \frac{17}{11} = \boxed{17:11}$

9.



Let BD produced and AC meet at F. Then join EF.

Denoting area of triangle ABC as $[ABC]$,

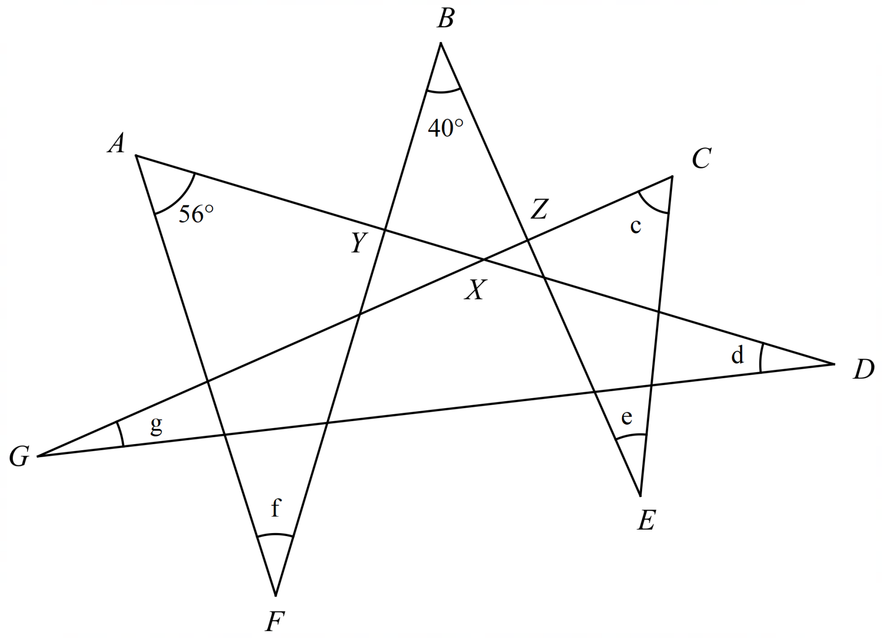
We see that $[ABD] = [BED] = u$ and $[ADF] = [DEF] = v$.

So, $u + v = 30$

Also, $[BEF] = \frac{3}{4}[ECF] = u + v$, i.e. $[\mathrm{ECF}] = \frac{4}{3}(u+v)$

$[ABC] = 2(u+v) + \frac{4}{3}(u+v) = \frac{10}{3}(30) = \boxed{100}$

10.



Label the angles according to their vertices C, D, E, F and G.

$\angle YXZ = \angle GXD = 180° - d - g$

$\angle BYX = \angle AYF = 180° - 56° - f$

$\angle BZX = \angle CZE = 180° - c - e$

Now, $40° + 180° - d - g + 180° - 56° - f + 180° - c - e = 360°$

$$c + d + e + f + g = 180° + 40° - 56° = \boxed{164°}$$

11.

$$
\begin{aligned}
N &= \frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + \ldots + \frac{1}{2024^2-1} \\
&= \frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \ldots + \frac{1}{2023\times 2025} \\
&= \frac{1}{2}\left(\frac{3-1}{1\times 3} + \frac{5-3}{3\times 5} + \frac{7-5}{5\times 7} + \ldots + \frac{2025-2023}{2023\times 2025}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \ldots + \frac{1}{2023} - \frac{1}{2025}\right) \\
&= \frac{1}{2}\left(1 - \frac{1}{2025}\right) \\
&= \frac{1012}{2025}
\end{aligned}
$$

Thus sum of digits of N is $\boxed{13}$.

12.

We have $a_4 \equiv 7^4$

$\equiv 2\times 7 \pmod{11}$

$\equiv 3 \quad \pmod{11}$

Then $a_5 \equiv 3\times 7 \equiv 10,\ a_6 = 4,\ a_7 = 6,\ a_8 = 9,\ a_9 = 8$ and $a_{10} = 1$

So the sequence repeats in cycles of 10. To find $a_1 + a_2 + a_3 + \ldots + a_{2024}$,

we note that $2024 = 10\times 202 + 4$.

Thus, $a_1 + a_2 + a_3 + \ldots + a_{2024} = (1+2+3+\ldots+10)\times 202 + 7+5+2+3$

$= 55\times 202 + 17$

$= \boxed{11127}$.

13.

$$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{31}\right)+\left(\frac{2}{3}+\frac{2}{4}+\frac{2}{5}+\ldots+\frac{2}{31}\right)+\left(\frac{3}{4}+\frac{3}{5}+\ldots+\frac{3}{31}\right)+\ldots+\left(\frac{29}{30}+\frac{29}{31}\right)+\frac{30}{31}$$

$$=\frac{1}{2}+\left(\frac{1}{3}+\frac{2}{3}\right)+\left(\frac{1}{4}+\frac{2}{4}+\frac{3}{4}\right)+\ldots+\left(\frac{1}{31}+\frac{2}{31}+\ldots+\frac{30}{31}\right)$$

$$=\frac{1}{2}+\frac{1}{3}(1+2)+\frac{1}{4}(1+2+3)+\ldots+\frac{1}{31}(1+2+\ldots+30)$$

$$=\frac{1}{2}+\frac{1}{3}\cdot\frac{2\times 3}{2}+\frac{1}{4}\cdot\frac{3\times 4}{2}+\ldots+\frac{1}{31}\cdot\frac{30\times 31}{2}$$

$$=\frac{1}{2}(1+2+3+\ldots+31)$$

$$=\frac{1}{2}\cdot\frac{31\times 32}{2}$$

$$=\boxed{232.5}$$

14.

Let Alice, Bob and Charlie each have a, b and c pens respectively.

$c+20 = 2(a-20+b) \quad \Rightarrow \quad c = 2a+2b-60$

$b+30 = 3(a-30+c) \quad \Rightarrow \quad b = 3a+3c-120$

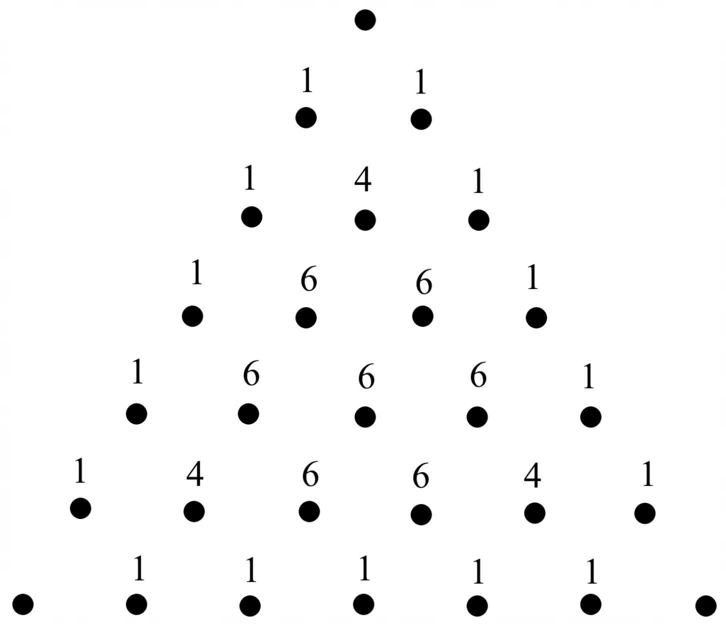
So, $b = 3a + 3(2a+2b-60) - 120$

$9a + 5b = 300$ and $a \geq 30$

The only admissible solution is $a = 30$ and $b = 6$.

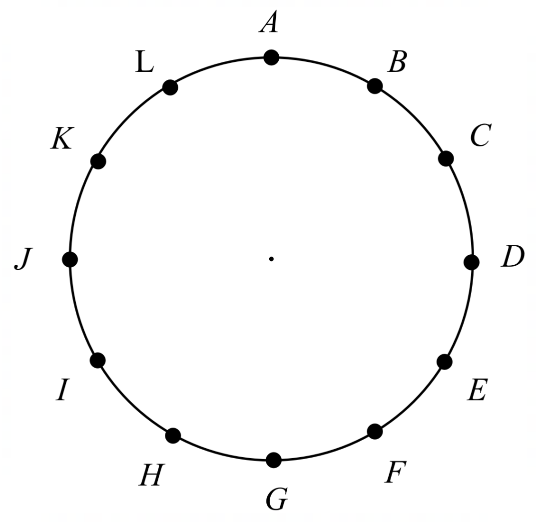
Thus, $c = 12$, and $a+b+c = \boxed{48}$.

15. The line segments can be classified by the midpoint dot, since then each line segment is counted only once. Also, the number of line segments through each dot form a symmetric patterns of numbers:



There are $\boxed{69}$ line segments.

16.



We label the points as shown and consider the cases by fixing the first vertex as *A*.

1^0 If the second vertex is *B*, then the third vertex can be *C*, *D*, *E*, *F* or *G*, giving us five different triangles.

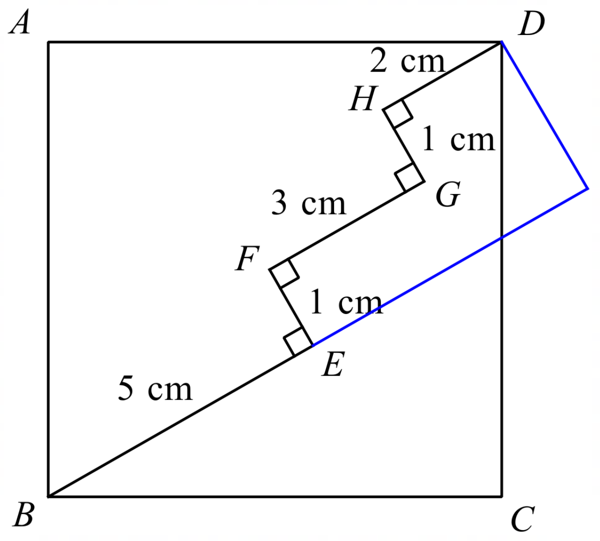
2^0 If the second vertex is *C*, then the third vertex can be *E, F, G or H*, giving us four different triangles.

3^0 If the second vertex is *D*, then the third vertex can be *G* or *H*, giving us two different triangles.

4^0 If the second vertex is *E*, then the third vertex can only be *I*, so there only 1 such triangle.

In total, we have $5 + 4 + 2 + 1 = \boxed{12}$ different triangles.

17.



$$BD^2 = (5+3+2)^2 + (1+1)^2$$
$$= 104$$
$$= \left(\sqrt{52}\right)^2 + \left(\sqrt{52}\right)^2$$

Thus, area of square is $\boxed{52 \text{ cm}^2}$.

18.

$x + \dfrac{y}{z} = 15$ and $y + \dfrac{x}{z} = 20$ where x, y and $z \in \mathbb{Z}^+$.

$\Rightarrow \quad (y-x) + \dfrac{x-y}{z} = 5$

$\Rightarrow \quad (y-x)\left(1 - \dfrac{1}{z}\right) = 5$

$\Rightarrow \quad (y-x)(z-1) = 5z$

Clearly, $z - 1 = 5$, so, $z = 6$.

Then, $y - x = 6$ or $x = y - 6$

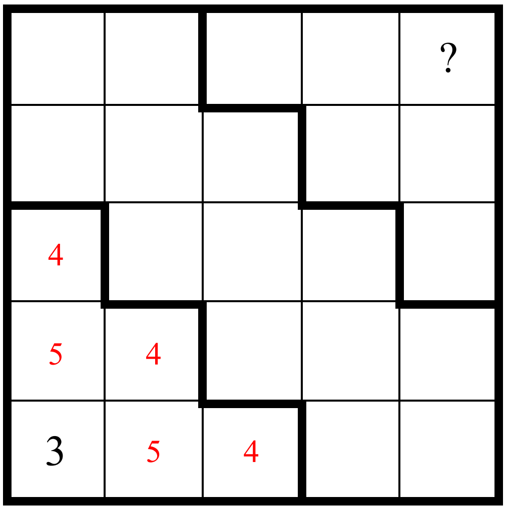
Using the first equation, $6x + y = 90$

Substituting: $6(y-6) + y = 90$

Thus, $y = 18$ and $x = 12$

and the sum $\dfrac{x+y}{z}$ is $\boxed{5}$.

19.



The sum for each block is $(1+2+3+4+5)\times 5\div 3=25$.

For the bottom block, the remaining entries cannot take 5, 5, 5, 4, 3.

So the only way is 5, 5, 4, 4, 4.

We have 4, 4 to put in the right top 2 x 2 grid, that means the remaining 4 entries in the top block must add up to 17.

Clearly, 5, 5, 5, 2 is the only way.

Consequently, the right top entry can only be $\boxed{2}$.

20.

$$\frac{[1,2]}{1\times 2}=\frac{2}{2}=1,\quad \frac{[1,2,3]}{1\times 2\times 3}=\frac{2\times 3}{2\times 3}=1,\quad \frac{[1,2,3,4]}{1\times 2\times 3\times 4}=\frac{2^2\times 3}{2\times 3\times 4}=\frac{1}{2},$$

$$\frac{[1,2,3,4,5]}{1\times 2\times 3\times 4\times 5}=\frac{2^2\times 3\times 5}{2\times 3\times 4\times 5}=\frac{1}{2},\quad \frac{[1,2,3,4,5,6]}{1\times 2\times 3\times 4\times 5\times 6}=\frac{2^2\times 3\times 5}{2\times 3\times 4\times 5\times 6}=\frac{1}{12},$$

$$\frac{[1,2,3,4,5,6,7]}{1\times 2\times 3\times 4\times 5\times 6\times 7}=\frac{2^2\times 3\times 5\times 7}{2\times 3\times 4\times 5\times 6\times 7}=\frac{1}{12},\ \ldots$$

From the calculations, we observe that

$\dfrac{[1,2,\ldots,n-1]}{1\times 2\times\ldots\times(n-1)}$ to $\dfrac{[1,2,\ldots,n]}{1\times 2\times\ldots\times n}$ changes in value if n is composite,

except for the initial prime 3. Since there are 25 primes less than 100,

thus, we have $99-25+1=\boxed{75}$ different values.