Raffles Mathematical Olympiad 2022
Round 1 (Solutions)

Date: 29 March 2022 **Duration: 1 hour**

This paper consists of 20 questions.
*For practice purpose, the multiple choice options are removed.

The marks allocation is as follows:

Question Number	Correct	Unanswered	Incorrect
1 to 10	4 marks	1 mark	0 mark
11 to 20	6 marks	1 mark	0 mark

1.

$$\left(1+\frac{1}{3}\right)\times\left(1+\frac{1}{4}\right)\times\left(1+\frac{1}{5}\right)\times\ldots\times\left(1+\frac{1}{n}\right)$$

$$=\frac{4}{3}\times\frac{5}{4}\times\frac{6}{5}\times\ldots\times\frac{n}{n-1}\times\frac{n+1}{n}$$

$$=\frac{n+1}{3}$$

$$\frac{n+1}{3}>2022$$

$$n+1>6066$$

$$\Rightarrow n>6065$$

Smallest n is 6066, so sum of digits of $n = \boxed{\mathbf{18}}$.

2.

$$\frac{23}{130}=\frac{13}{130}+\frac{10}{130}=\frac{1}{10}+\frac{1}{13}$$

$$=0.1+0.076923076923076923.......$$

$$=0.1769230769230769230......$$

Since $2022=1+6\times336+5$, the 2022^{th} digit is $\boxed{\mathbf{3}}$.

3.

$$\overline{123abc789}$$

$$=123\times1000000+\overline{abc}\times1000+789$$

$$=123\times(999999+1)+\overline{abc}\times(999+1)+789$$

$$=\left(123\times999999+\overline{abc}\times999\right)+\left(123+\overline{abc}+789\right)$$

For the number to be divisible by 999, $\overline{abc}=999-789-123=\overline{087}$,

so $a+b+c=\boxed{\mathbf{15}}$.

4.

$$\frac{2022}{674+674^2}+\frac{2022}{675+675^2}+\frac{2022}{676+676^2}+\ldots+\frac{2022}{1010+1010^2}$$

$$=2022\left(\frac{1}{674\times675}+\frac{1}{675\times676}+\frac{1}{676\times677}+\ldots+\frac{1}{1010\times1011}\right)$$

$$=2022\left(\frac{1}{674}-\frac{1}{675}+\frac{1}{675}-\frac{1}{676}+\frac{1}{676}-\frac{1}{677}+\ldots+\frac{1}{1010}-\frac{1}{1011}\right)$$

$$=2022\left(\frac{1}{674}-\frac{1}{1011}\right)=3-2=\boxed{\mathbf{1}}.$$

5.

$$\frac{8088}{24}+\frac{8088}{40}+\frac{8088}{60}+\frac{8088}{84}+\frac{8088}{112}+\frac{8088}{144}+\frac{8088}{180}+\frac{8088}{220}+\frac{8088}{264}$$
$$=2022\left[\frac{1}{6}+\left(\frac{1}{10}+\frac{1}{15}\right)+\left(\frac{1}{21}+\frac{1}{28}\right)+\left(\frac{1}{36}+\frac{1}{45}\right)+\left(\frac{1}{55}+\frac{1}{66}\right)\right]$$
$$=2022\left[\frac{1}{6}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}\right]$$
$$=2022\left[\frac{1}{6}+\frac{1}{6}+\frac{1}{12}+\frac{1}{12}\right]$$
$$=\boxed{\mathbf{1011}}.$$

6.

Ratio of diameters $=1:2:3$

Ratio of areas of circles $=1:4:9=S:4S:9S$

Area of unshaded parts in cm^2 $=9S-4S-S+S+S=6S=6\times\frac{22}{7}\times\left(\frac{7}{2}\right)^2=\boxed{\mathbf{231}}$.

7.

$$N=2022+\frac{2022}{1+2}+\frac{2022}{1+2+3}+\frac{2022}{1+2+3+4}+\ldots+\frac{2022}{1+2+3+4+\ldots+2021}$$
$$=2022\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\ldots+\frac{1}{1+2+3+4+\ldots+2021}\right)$$
$$=2022\left[2\left(\frac{1}{1}-\frac{1}{2}\right)+2\left(\frac{1}{2}-\frac{1}{3}\right)+2\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots.+2\left(\frac{1}{2021}-\frac{1}{2022}\right)\right]$$
$$=2022\left[2\left(1-\frac{1}{2022}\right)\right]$$
$$=4044-2$$
$$=4042$$

Sum of digits $=4+4+2=\boxed{\mathbf{10}}$.

8.

$$\text{Sum}=30\times 2022+\frac{6}{2}(5+7+9+11+13)=\boxed{\mathbf{60795}}.$$

9.

Let x be the number of male contestants in 2015.
Let y be the number of female contestants in 2015.

In 2016,

No. of male contestants = $\frac{6}{5}x$

No. of female contestants = $\frac{9}{5}y$

$$\frac{6}{5}x+\frac{9}{5}y=\frac{33}{25}(x+y)$$
$$\frac{3}{5}x=\frac{12}{5}y$$
$$x=4y$$

Required fraction $=\dfrac{\frac{9}{5}y}{\frac{6}{5}x+\frac{9}{5}y}=\dfrac{\frac{9}{5}y}{\frac{6}{5}(4y)+\frac{9}{5}y}=\dfrac{9}{33}=\boxed{\mathbf{\frac{3}{11}}}$.

10.

Let Mary's speed be x km/h.

Let the perimeter of the track be y km.

Suppose both of them ride anticlockwise: $(36-x)\times\frac{80}{60}=\frac{y}{2}$

Suppose Esther rides anticlockwise and Mary rides clockwise: $(36+x)\times\frac{10}{60}=\frac{y}{2}$

$$(36-x)\times\frac{80}{60}=(36+x)\times\frac{10}{60}$$
$$288-8x=36+x$$
$$9x=252$$
$$x=28$$

So, perimeter = $y=\frac{64}{3}=\boxed{\mathbf{21\frac{1}{3}\ km}}$.

11.

Let the number of litres of water drank per person be *m* and the number of days the amount of water distributed can last be *n*.

$mn = (m-1)(n+15) \quad ......(1)$

$mn = (m-1.5)(n+30)......(2)$

From (1): $15m - n = 15$

From (2): $30m - 1.5n = 45$

Solving, $\qquad 0.5n = 15$

$\qquad\qquad \Rightarrow n = 30$

Therefore, the water distributed can last **30 days**.

12.

Eric wears yellow clothes on odd days and Jasmine wears yellow when the date is a multiple of 3.
So both of them wear yellow clothes on 3^{rd}, 9^{th} , 15^{th}, 21^{st} and 27^{th} day of each month, and the number of days when both wear clothes of different colours is given by
$365 - 5 \times 12 =$ **305 days**.

13.

X		X
	X	
X		X

Among the first 9 integers, 5 odd integers and 4 even integers. Clearly to satisfy the requirement, we need to fill up the cells labelled "X" with odd integers and the rest with even integers.

Number of ways = $(5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1) = 120 \times 24 =$ **2880**.

14.

Divide the 30 consecutive positive integers into 3 groups.

Group 1 : When divided by 3, remainder is 0
Group 2 : When divided by 3, remainder is 1
Group 3 : When divided by 3, remainder is 2

Each group has 10 numbers.

For the sum of the 3 numbers to be divisible by 3, the 3 numbers must be chosen in the following ways :

Case 1 : all 3 numbers from group 1, 2 or 3

$$\text{No of ways} = \binom{10}{3} \times 3$$

Case 2 : 1 number from each group

$$\text{No of ways} = \binom{10}{1} \times \binom{10}{1} \times \binom{10}{1}$$

Therefore the number of ways $= \binom{10}{3} \times 3 + \binom{10}{1} \times \binom{10}{1} \times \binom{10}{1} = 360 + 1000 = \boxed{\mathbf{1360}}$.

15.

Let the original height of water by x cm.
Let the lengths of rulers B and C be b cm and c cm respectively.
After adding some water, the height of water be $2x$ cm.

From the given information,

$$20 - x : b - x : c - x = 1:2:4$$

$$20 - 2x : b - 2x : c - 2x = 1:3:7$$

$$\frac{20 - x}{c - x} = \frac{1}{4}$$

$$80 - 4x = c - x$$

$$c = 80 - 3x$$

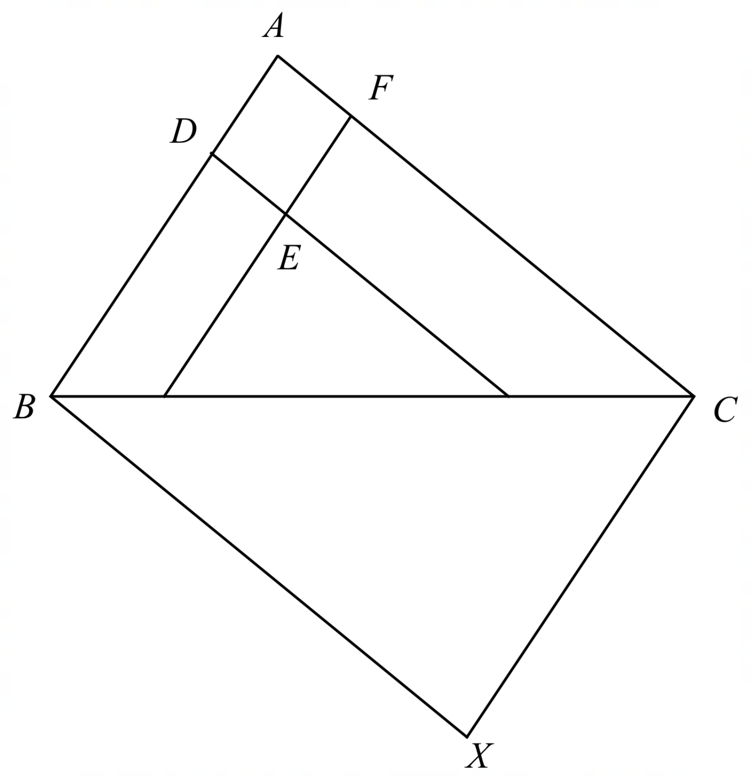
$$\frac{20 - 2x}{c - 2x} = \frac{1}{7}$$

$$140 - 14x = c - 2x$$

$$140 - 14x = 80 - 3x - 2x$$

$$x = \frac{20}{3} = \boxed{\mathbf{6\frac{2}{3}}}.$$

16.



Let X be the point such that BX is parallel to AC and CX is parallel to AB.
Then the parallelograms $ABXC$ and $ADEF$ have equal corresponding angles i.e. they are similar.
Hence area of triangle ABC

$$=\frac{1}{2}\times\frac{AB}{AD}\times\frac{AC}{AF}\times(\text{area of parallelogram } ADEF)$$

$$=\frac{1}{2}\times\frac{26}{7}\times\frac{33}{6}\times 14$$

$$=\boxed{\mathbf{143\ cm^2}}.$$

17.

Let $S=\frac{1}{5}+\frac{1}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\frac{5}{5^5}+\frac{8}{5^6}+\ldots\quad(1)$

Then $\frac{S}{5}=\frac{1}{5^2}+\frac{1}{5^3}+\frac{2}{5^4}+\frac{3}{5^5}+\frac{5}{5^6}+\frac{8}{5^7}+\ldots\quad(2)$

$(1)-(2)$: $\frac{4}{5}S=\frac{1}{5}+\left(\frac{1}{5^3}+\frac{1}{5^4}+\frac{2}{5^5}+\frac{3}{5^6}+\ldots\right)$

$$=\frac{1}{5}+\frac{1}{5^2}\left(\frac{1}{5}+\frac{1}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\ldots\right)$$

$$\frac{4}{5}S=\frac{1}{5}+\frac{1}{25}S$$

Thus, $S=\boxed{\mathbf{\frac{5}{19}}}$.

18.

Let $\angle BCA = p°$ and $\angle FCA = q°$

$p + q = 46$

$\angle CAD = p°$ (alternate $\angle$s)

Since $AC = AD$, $\angle ACD = \dfrac{180° - p°}{2}$

Since $AC = CF$, $\angle CAF = \dfrac{180° - q°}{2} = \angle ECA$ (alt $\angle$s)

$$\angle ECD = \frac{180° - p°}{2} + \frac{180° - q°}{2} = \frac{360° - (p° + q°)}{2} = \frac{360° - 46°}{2} = \boxed{\mathbf{157°}}.$$

19.

For a child with label *n*, we consider the factors of *n*.
For each odd factor, 3 candies are added and for each even factor, 1 candy is taken away. So the child has net addition of candies if the number of odd factors exceeds 3 times the number of even factors of *n* and vice versa.

Write $n = 2^r s$, s : odd.
So every odd factor of *n* corresponds to *r* even factors of *n*.
Hence the child with label *n* gets extra candies if and only if $r < 3$ ie. *n* is not divisible by 8. Since there are 125 multiples of 8 among the first 1000 positive integers, the answer is $1000 - 125 = \boxed{\mathbf{875}}$.

20.

Let the number of \$1.4, \$1.8, \$2.4 and \$3 be a, b, c and d respectively.

To solve $1.4a + 1.8b + 2.4c + 3d = 12$

i.e. $7a + 9b + 12c + 15d = 60$

where a, b, c, d are non negative integers.

Since all the coefficients are divisible by 3 except 7, a must be divisible by 3.
So possible values for a are 0, 3 or 6.

$1°$: $a = 6$. Then equation reduces to $3b + 4c + 5d = 6$

There is only 1 solution: $b = 2, c = d = 0$.

$2°$: $a = 3$. Then equation reduces to $3b + 4c + 5d = 13$

The possible solutions are:

$d = 2, b = 1, c = 0$

$d = 1, c = 2, b = 0$

$d = 0, b = 3, c = 1$

$3°$: $a = 0$. Then equation reduces to $3b + 4c + 5d = 20$

The possible solutions are:

$d = 4, b = 0, c = 0$

$d = 3$ (no solutions)

$d = 2, c = 1, b = 2$

$d = 1, b = 1, c = 3$

$d = 1, b = 5, c = 0$

$d = 0, b = 0, c = 5$

$d = 0, b = 4, c = 2$

In total, there are **10 ways**.