Raffles Mathematical Olympiad 2023
Round 1 (Solutions)

Date: 28 March 2023 **Duration: 1 hour**

This paper consists of 20 questions.
*For practice purpose, the multiple choice options are removed.

The marks allocation is as follows:

Question Number	Correct	Unanswered	Incorrect
1 to 10	4 marks	1 mark	0 mark
11 to 20	6 marks	1 mark	0 mark

1. We have

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)>\frac{1823}{2023}$$

$$\Rightarrow \quad 1-\frac{1}{n+1}>\frac{1823}{2023}$$

$$\Rightarrow \quad \frac{1}{n+1}<\frac{200}{2023}$$

$$\Rightarrow \quad n>\frac{1823}{200}=9\frac{23}{200}$$

$\therefore$ least integer $n=\boxed{\mathbf{10}}$.

2. When the number is multiplied to itself, we have

$$\underbrace{999\ldots99}_{\text{2023 digits}}\times\underbrace{999\ldots99}_{\text{2023 digits}}$$

$$=\underbrace{999\ldots99}_{\text{2023 digits}}\times\left(1\underbrace{000\ldots00}_{\text{2023 digits}}-1\right)$$

$$=\underbrace{999\ldots99}_{\text{2023 digits}}\underbrace{000\ldots00}_{\text{2023 digits}}-\underbrace{999\ldots99}_{\text{2023 digits}}$$

$$=\underbrace{999\ldots99}_{\text{2022 digits}}8\underbrace{000\ldots00}_{\text{2022 digits}}1$$

Thus the sum is $9\times 2022+8+1=\boxed{\mathbf{18207}}$.

3.

Let $a=\frac{1}{17}+\frac{2}{39}+\frac{3}{53}+\frac{4}{79}+\frac{5}{98}$, $b=\frac{2}{39}+\frac{3}{53}+\frac{4}{79}+\frac{5}{98}$.

Then $a-b=\frac{1}{17}$.

Now, required sum

$$=a\left(b+\frac{6}{119}\right)-\left(a+\frac{6}{119}\right)b$$

$$=ab+\frac{6a}{119}-ab-\frac{6b}{119}$$

$$=\frac{6}{119}(a-b)$$

$$=\boxed{\mathbf{\frac{6}{2023}}}.$$

4. Since $\underbrace{20232023...2023}_{k \text{ copies of } 2023}1965$ is already a multiple of 5 and is divisible by 55, then it must also be divisible by 11.

Taking difference of the sum of digits in odd and even places, we have

$$\left[(2+2)k+1+6\right]-\left[(0+3)k+9+5\right]$$

$$= k-7 \text{ which must be zero or a multiple of } 11$$

Thus, least k is $\boxed{7}$.

5. Let $X = \dfrac{1}{\dfrac{1}{2013}+\dfrac{1}{2014}+\dfrac{1}{2015}+\cdots+\dfrac{1}{2023}}$

$$\frac{1}{\dfrac{1}{2013}+\dfrac{1}{2013}+\dfrac{1}{2013}+\cdots+\dfrac{1}{2013}} < X < \frac{1}{\dfrac{1}{2023}+\dfrac{1}{2023}+\dfrac{1}{2023}+\cdots+\dfrac{1}{2023}}$$

$$\frac{2013}{11} < X < \frac{2023}{11}$$

i.e. $$183 < X < 183\frac{10}{11}$$

Thus, the integer part is $\boxed{\mathbf{183}}$.

6. Suppose Jack has x green marbles, so he'll have $4x$ blue marbles, and kx red marbles, where k is a positive integer.

Thus, we get $4x + 22 = 5kx$.

$$(5k-4)x = 22$$

From here, we deduce that $5k-4$ is a factor of 22, so $5k-4$ is either 1, 2, 11 or 22.

Since k is a positive integer, we have $5k-4 = 1$ or 11, whereupon $k = 1$ or 3.

If $k = 1$, then $x = 22$ and $4x = 88$ is a contradiction.

So, when $k = 3$, we have $x = 2$, $4x = 8$ and $kx = 6$, thus the number of marbles that Jack owns which are neither blue, green, nor red is $40-2-8-6 = \boxed{\mathbf{24}}$.

7. Suppose Bob's monthly income is $4x$ while Charlie's $3x$. Then we have

$$\frac{4x-\dfrac{9000}{12}}{3x-\dfrac{9000}{12}} = \frac{11}{7}$$

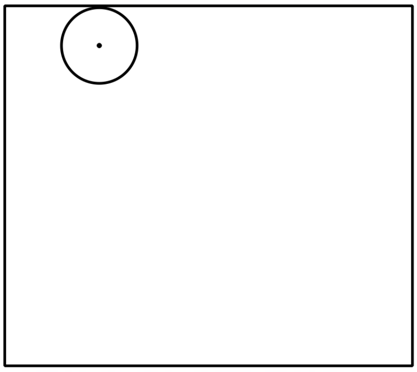
$$7(4x-750) = 11(3x-750)$$

$$33x-28x = 750(11-7)$$

$$x = 150(4) = 600$$

Thus, Bob's monthly income = $\boxed{\mathbf{\$2400}}$.

8.



Total area covered by the circle

$=10^2 - 6^2 - \left[2^2 - 3.14\times 1^2\right]$

$= \boxed{\mathbf{63.14\ cm^2}}$.

9. $a\left(b^c c^d + 1\right) = ab^c c^d + a = 2023 = 7.17^2$

Thus, $a = 7$.

Next $17^2 = 289$

$= 288 + 1$

$= 2^5.3^2 + 1$

$= 3^2.2^5 + 1$

Thus, $b = 3,\ c = 2$ and $d = 5$.

Hence, required sum $= 7.3 + 3.2 + 2.5 + 5.7$

$= \boxed{\mathbf{72}}$.

10. Taking the total scores over k days, we have

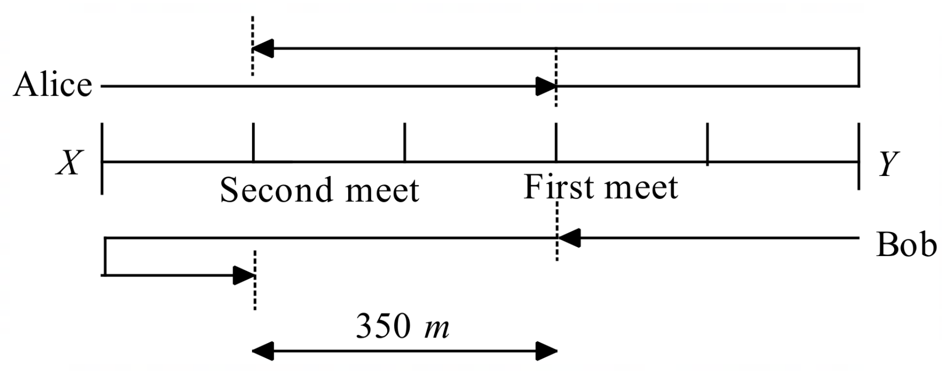
$k\left(1+2+3+\cdots+n\right) = 40n$

$\frac{kn\left(n+1\right)}{2} = 40n$

$k\left(n+1\right) = 80 = 1.80 = 2.40 = 4.20 = 5.16$

Since k is a factor of n, and $k > 1$, we have $k = 5$ and $n = \boxed{\mathbf{15}}$.

11. Since Alice's speed is 1.5 times that of Bob's, in other words, for every 3 units covered by Alice, Bob would walk 2 units, thus we can partition the distance between X and Y into 5 units as shown in the figure:

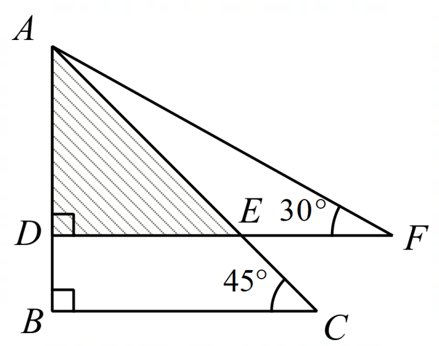


Distance of $XY = 350 \times \frac{5}{2} =$ **875 m**.

12. There is a total of 420 males and 580 females (i.e. 1000 people) in both villages.
After the move of some villagers from B to A,
ratio of males to females in
Village A is 2u: 3u and Village B is 3v: 2v, where u > v
We thus have $5u + 5v = 1000$ ………. (1)
and $3u + 2v = 580$ ………... (2)
Solving these equations: $v = 20$, $u = 180$
Thus, a total of $400 - 5(20) =$ **300 people** moved from B to A .

13. Least number of students who can play all four sports
$= 46 - [(46 - 40) + (46 - 38) + (46 - 35) + (46 - 27)]$
= **2 students**

14.



Using the labelled diagram, we note that AD = DE, so

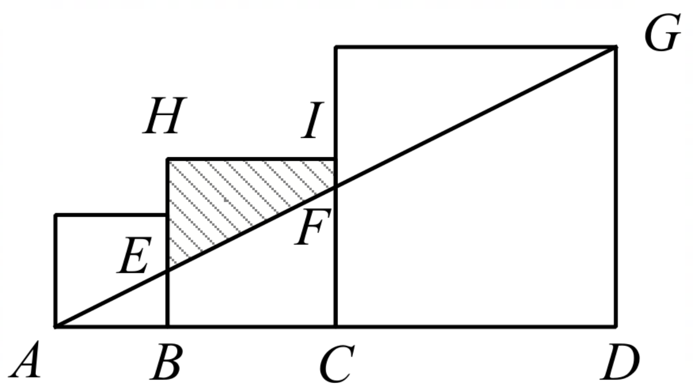
$\frac{1}{2} AD^2 = 578$

$AD^2 = 2^2 . 17^2$

$AD = 34$ cm

We have AD = ½ AF, thus AF = **68 cm**.

15.



Using the labelled diagram,

$$\frac{BE}{DG}=\frac{AB}{AD}=\frac{3}{3+5+8} \Rightarrow BE=\frac{3}{16}\times 8=1.5 \text{ cm}$$

$$\Rightarrow EH=3.5 \text{ cm}$$

$$\frac{CF}{BE}=\frac{AC}{AB}=\frac{3+5}{3} \Rightarrow CF=\frac{8}{3}\times\frac{3}{2}=4 \text{ cm}$$

$$\Rightarrow FI=1 \text{ cm}$$

$$\text{Area of shaded region}=\frac{1}{2}(3.5+1)(5)$$

$$=\boxed{\mathbf{11.25\,cm^2}}$$

16. Since 6 can be prime factorised as 2.3, we can consider all possible marks are of the form $3x + 4y$, where $0 \le x \le 20$, $0 \le y \le 10$. By varying y, we consider cases of getting possible total marks.

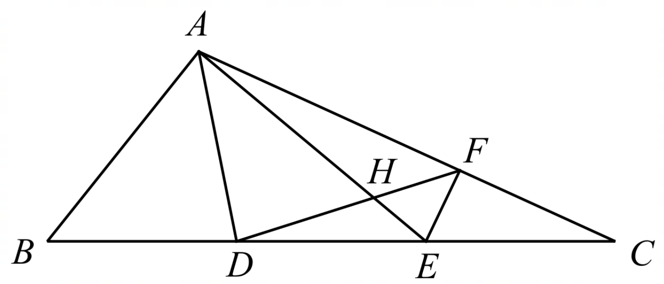
If $y = 0$, possible total marks between (*and inclusive* henceforth) 0 to 60 are multiples of 3.
If $y = 1$, possible total marks between 4 to 64 are 1 + multiples of 3
If $y = 2$, possible total marks between 8 to 68 are 2 + multiples of 3
If $y = 3$, possible total marks between 12 to 72 are multiples of 3
If $y = 4$, possible total marks between 16 to 76 are 1 + multiples of 3
If $y = 5$, possible total marks between 20 to 80 are 2 + multiples of 3
If $y = 6$, possible total marks between 24 to 84 are multiples of 3
If $y = 7$, possible total marks between 28 to 88 are 1 + multiples of 3
If $y = 8$, possible total marks between 32 to 92 are 2 + multiples of 3
If $y = 9$, possible total marks between 36 to 96 are multiples of 3
If $y = 10$, possible total marks between 40 to 100 are 1 + multiples of 3

For total marks which are
1 + multiples of 3 between 0 to 96, it's impossible to obtain 1 mark;
2 + multiples of 3 between 8 to 92, it's impossible to obtain 2, 5, 95, 98 marks
Multiples of 3 between 0 to 96, it's impossible to obtain 99 marks.

In all, these **6 scores**: 1, 2, 5, 95, 98 and 99 are not possible.

17. Since each supermarket would basically have 4 boxes of apples and 3 boxes of oranges delivered, we would only need to consider the no. of ways to distribute the remaining 3 boxes of apples and 3 boxes of oranges to these five supermarkets. For the 3 boxes of apples, they can be distributed in $\binom{5}{1}+\binom{5}{1}\binom{4}{1}+\binom{5}{3}=5+20+10=35$ ways, based on whether they are distributed to 1, 2 or 3 supermarket(s) respectively. Similarly, there are 35 ways to distribute the 3 boxes of oranges. Since these two are independent distributions, the total no. of ways to distribute the fruits are 35.35 = **1225 ways**.

18.



Let's use [ABC] to denote the area of triangle ABC.
3[ADE] = [ABC] and [ECF] = [EDF].
Since $\Delta ECF \sim \Delta BCA$, we have 9[ECF] = [ABC].

Moreover [ADH] – [EFH] =12.6 $\Rightarrow$ [ADE] – [EDF] = 12.6
1/3 [ABC] – 1/9 [ABC] = 12.6
2/9 [ABC] = 12.6
Thus, [ABC] = 12.6 (9/2) = **56.7 cm²** .

19.

Row No.	Pattern	Sum
1	1, 2	3
2	1, 3, 2	3 + 3
3	1, 4, 3, 5, 2	3 + 3 + 9
4	1, 5, 4, 7, 3, 8, 5, 7, 2	3 + 3 + 9 + 27
5	1, 6, 5, 9, 4, 11, 7, 10, 3, 11, 8, 13, 5, 12, 7, 9, 2	3 + 3 + 9 + 27 + 81

By the pattern established for the sum, we have by row 9, sum of entries

$= 3+3^1+3^2+3^3+\ldots+3^8$

$= 3+\dfrac{3(3^8-1)}{3-1}$

$= 3+\dfrac{3(6560)}{2}$

$=$ **9843**.

20.

Suppose we have $a > b > c > d$ and $d \neq 0$.

We will demonstrate that this case cannot work.

So the largest number is $\overline{abcd}$ and the smallest is $\overline{dcba}$.

Then, $\overline{abcd} + \overline{dcba} = 11359$ is not possible as we run through all cases of $d = 1,\ 2,\ ...,\ 9$, and corresponding $a = 8,\ 7,\ 6,\ ...,\ 0$.

Thus, we would consider $a > b > c > d$ and $d = 0$.

So the largest number is $\overline{abc0}$ and the smallest is $\overline{c0ba}$.

Now, from $\overline{abc0} + \overline{c0ba} = 11359,$ we can deduce that $a = 9$.

Further, $a + c = 11$ tells us that $c = 2$ and so $b = 3$.

Then the difference between the two numbers 9320 and 2039 is $\boxed{\mathbf{7281}}$.